Section 14.8

Lagrange Multipliers

1 Lagrange Multipliers, Identifying Extrema on Boundaries

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Constrained Optimization (Handout Only)

A typical constrained optimization problem:

Find the maximum value of
$f(x,y) = xy$
subject to the constraint $x^2 + 4y^2 = 16$**.**

- Effectively, the domain of f consists of all points satisfying the constraint equation.

  (Here the constraint defines an ellipse E.)

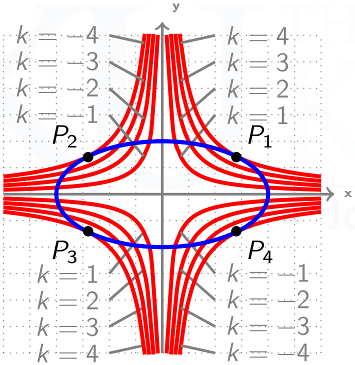
- We cannot simply find and test the critical points, since they might not satisfy the constraint.

  (Here $\nabla f(x,y) = \langle y, x \rangle$, so the unique critical point is $(0,0)$, which is not on E.)

- However, the domain is closed and bounded, so f must attain an absolute minimum and maximum by the Extreme Value Theorem.

Constrained Optimization (Handout Only)

Example 1: Find the maximum value of $f(x, y) = xy$ subject to the constraint $x^2 + 4y^2 = 16$.



The Level Curves of $f(x, y) = xy$

Link

The constraint curve is tangent to the level curves $f(x, y) = 4$ at P_1 and P_3, where f attains maximum value on the constraint.

The constraint curve is tangent to the level curves $f(x, y) = -4$ at P_2 ad P_4, where f attains minimum value on the constraint.

Therefore, $\nabla f(x, y)$ and $\nabla(x^2 + 4y^2)$ are parallel at P_1, P_2, P_3 and P_4.

Example 1, Continued (Handout Only)

<u>Solution:</u>

Set $\langle y, x\rangle = \lambda\langle 2x, 8y\rangle$ for x and y values of P_1, P_2, P_3 and P_4.

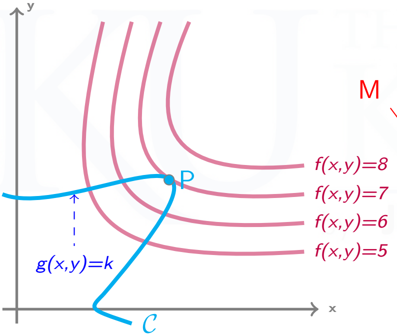
$$\begin{cases} y &= \lambda(2x) \\ x &= \lambda(8y) \\ 16 &= x^2 + 4y^2 \quad \text{(The third equation)} \end{cases} \quad \Longrightarrow \quad \begin{matrix} x^2 = 4y^2 \\ \\ x^2 + 4y^2 = 16 \end{matrix}$$

Solve: $x = \pm 2\sqrt{2}$ and $y = \pm\sqrt{2}$. Points are $(\pm 2\sqrt{2}, \pm\sqrt{2})$.

(x, y)	$P_1(2\sqrt{2}, \sqrt{2})$	$P_2(-2\sqrt{2}, \sqrt{2})$	$P_3(-2\sqrt{2}, -\sqrt{2})$	$P_4(2\sqrt{2}, -\sqrt{2})$
$f(x, y)$ Classification:	$f(2\sqrt{2}, \sqrt{2}) = 4$ Max	$f(-2\sqrt{2}, \sqrt{2}) = -4$ Min	$f(-2\sqrt{2}, -\sqrt{2}) = 4$ Max	$f(2\sqrt{2}, -\sqrt{2}) = -4$ Min

Constrained Optimization

Objective: Find the extrema of $z = f(x, y)$ subject to the constraint $g(x, y) = k$, where k is a constant.



$g(x, y) = k$ is a constraint curve $\mathcal{C}$.

Suppose $P(a, b)$ is a local extremum of f on $\mathcal{C}$ and $f(a, b) = M$.

At (a, b), the level curve $f(x, y) = M$ is **tangent** to $\mathcal{C}$.

- Since $\mathcal{C}$ is a level curve of $g(x, y)$, $\nabla g(a, b)$ is orthogonal to $\mathcal{C}$.
- Since $f(x, y) = M$ is tangent to $\mathcal{C}$ at P, $\nabla f(a, b)$ is orthogonal to $\mathcal{C}$.
- Therefore, $\nabla f(a, b)$ is parallel to $\nabla g(a, b)$.

Lagrange Multipliers

Assume that $f(x, y)$ and $g(x, y)$ are differentiable functions.

If f has a local extremum on the constraint curve $g(x, y) = k$ at $P(a, b)$, and if $\nabla g(a, b) \neq \vec{0}$, then there exists a scalar λ such that

$$\nabla f(a, b) = \lambda \nabla g(a, b)$$

Proof (Optional): Let $\vec{r}(t)$ parametrize $g(x, y) = k$ near P, chosen such that $\vec{r}(t_0) = (a, b)$ and $\vec{r}\,'(t_0) \neq \vec{0}$.

By assumption, $f(\vec{r}(t))$ is a local extremum of f at $t = t_0$.

Thus, $t = t_0$ is a critical point of $f(\vec{r}(t))$ and by the Chain Rule

$$\left.\frac{d}{dt} f(\vec{r}(t))\right|_{t=t_0} = \underbrace{\nabla f(a, b)}_{\nabla f(\vec{r}(t_0))} \cdot \vec{r}\,'(t_0) = 0$$

Therefore, $\nabla f(a, b)$ is orthogonal to $\vec{r}\,'(t_0)$. Meanwhile, since $\nabla g(a, b)$ is orthogonal to its level curve $\vec{r}(t)$ at $t = t_0$, $\nabla g(a, b)$ is also orthogonal to $\vec{r}\,'(t_0)$. Therefore, $\nabla f(a, b)$ is parallel to $\nabla g(a, b)$. ▸ Video

The Method of Lagrange Multipliers

To find the extrema of $z = f(x, y)$ subject to the constraint $g(x, y) = k$,

(1) Find all values a, b, and λ such that $g(a, b) = k$ and

$$\nabla f(a, b) = \lambda \nabla g(a, b)$$

- There are a total of three equations and three unknowns.
- Often, the best way to solve this system is to start by eliminating λ.

(2) Calculate the values of f at all points (a, b) found in step (1) and the end points.

(3) The largest of these is the absolute maximum value and the smallest is the absolute minimum value of f constrained by $g(x, y) = k$ if the curve of the constraint is closed and bounded.

An analogous method works for functions of three or more variables. Instead of end points, we use boundary points.

Lagrange Multipliers: 2 Variables

Example 2: Find the extrema of $f(x,y) = x^2 - 4xy + y^2$ subject to the constraint $g(x,y) = x^2 + y^2 = 1$.

<u>Solution:</u> The gradients of f and g are

$$\begin{aligned} \nabla f(x,y) &= \langle 2x - 4y,\ -4x + 2y \rangle \\ \nabla g(x,y) &= \langle 2x,\ 2y \rangle \end{aligned}$$

So the Lagrange equations

$$\nabla f(x,y) = \lambda \nabla g(x,y), \qquad g(x,y) = 1$$

become

$$\begin{cases} 2x - 4y = 2\lambda x \\ -4x + 2y = 2\lambda y \\ x^2 + y^2 = 1 \end{cases} \qquad \text{or} \qquad \begin{cases} x - 2y = \lambda x \\ -2x + y = \lambda y \\ x^2 + y^2 = 1. \end{cases}$$

Lagrange Multipliers: 2 Variables

Example 2 (cont'd): Eliminating λ from the first two equations produces

$$\frac{x-2y}{x} = \frac{-2x+y}{y} \qquad \therefore \qquad y(x-2y) = x(-2x+y)$$

and simplifying produces $x^2 = y^2$. Since $x^2 + y^2 = 1$:

$$x^2 = y^2 = \frac{1}{2} \qquad \therefore \qquad x = \pm\frac{1}{\sqrt{2}}, \quad y = \pm\frac{1}{\sqrt{2}}$$

There are four critical points.

(x, y)	$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	$\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$	$\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	$\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$
$f(x, y)$ Classification	-1 Minimum	-1 Minimum	3 Maximum	3 Maximum

Lagrange Multipliers: 3 Variables

Example 3: Which points on surface $x^2y^2z^2 = 1$ are closest to the origin?

Solution: Minimizing distance $\sqrt{x^2 + y^2 + z^2}$ is equivalent to minimizing $f(x, y, z) = x^2 + y^2 + z^2$, and the algebra is easier.

The constraint is $g(x, y, z) = 1$, where $g(x, y, z) = x^2y^2z^2$.

System of equations to solve:

$$\begin{aligned}
2x &= 2\lambda xy^2z^2 && \text{(from } \nabla f(x,y,z) = \lambda \nabla g(x,y,z)\text{)} \\
2y &= 2\lambda x^2yz^2 && \text{(ditto)} \\
2z &= 2\lambda x^2y^2z && \text{(ditto)} \\
x^2y^2z^2 &= 1 && \text{(constraint equation)}
\end{aligned}$$

Note that the constraint is not bounded: Link

Lagrange Multipliers: 3 Variables

Example 3 (cont'd): The constraint equation implies $x, y, z \neq 0$, so we can eliminate λ from the first three to get

$$1/\lambda = y^2z^2 = x^2z^2 = x^2y^2.$$

which implies $x^2 = y^2 = z^2$. Since $x^2y^2z^2 = 1$ we conclude that

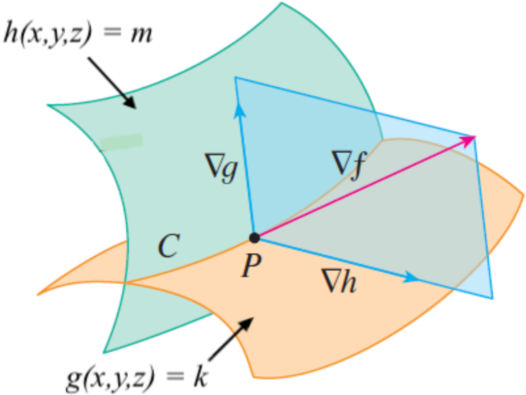
$$(x, y, z) = (\pm1, \pm1, \pm1).$$

Answer: Since $f(\pm1, \pm1, \pm1) = 3$, the closest distance is $\sqrt{3}$, which is attained at the eight points $(\pm1, \pm1, \pm1)$.

2 Multiple Lagrange Multipliers

Lagrange Multipliers with Two Constraints

What if we want to find the extrema of $f(x, y, z)$ subject to **two** constraints $g(x, y, z) = k$ and $h(x, y, z) = m$?



Lagrange Multipliers with Two Constraints

- The two constraints together define a curve C (the intersection of two surfaces)
- If P is a local extremum of f, then the tangent line to C at P should lie in the tangent plane to the level surface of f at P.
- ∇f is orthogonal to the tangent plane to the level surface of f.
- That tangent plane contains both ∇g and ∇h (because P is an extremum).
- Therefore, ∇f is a linear combination of ∇g and ∇h:

$$\nabla f = \lambda \nabla g + \mu \nabla h.$$

Lagrange Multipliers with Two Constraints

To find the extrema of $f(x, y, z)$ subject to two constraints $g(x, y, z) = k$ and $h(x, y, z) = m$:

(1) Solve the system of five equations and five unknowns:

$$\begin{aligned} \nabla f(a, b, c) &= \lambda \nabla g(a, b, c) + \mu \nabla h(a, b, c) \\ g(a, b, c) &= k \\ h(a, b, c) &= m \end{aligned}$$

(2) Calculate the values of f at all points (a, b, c) found in step (1) and the end points.

(3) The largest of these is the absolute maximum value and the smallest is the absolute minimum value if the intersection of the two constraints is closed and bounded.

An analogous method works for any numbers of variables and constraints.

Lagrange Multipliers with Two Constraints

Example 4: The temperature at (x, y, z) is $T(x, y, z) = 2x + 5y + 7z$. On the curve of intersection of $x - y + z = 1$ and $x^2 + y^2 = 1$, calculate the maximum and minimum temperatures.

Solution:

$$T(x, y, z) = 2x + 5y + 7z \qquad \nabla T = \langle 2, 5, 7 \rangle$$
$$g(x, y, z) = x - y + z \qquad \nabla g = \langle 1, -1, 1 \rangle$$
$$h(x, y, z) = x^2 + y^2 \qquad \nabla h = \langle 2x, 2y, 0 \rangle$$

System of equations to solve:

$$\begin{aligned} 2 &= \lambda + 2\mu x & x - y + z &= 1 \\ 5 &= -\lambda + 2\mu y & x^2 + y^2 &= 1 \\ 7 &= \lambda \end{aligned}$$

Eliminating λ and μ in the first three equations gives $-12x = 5y$.

Lagrange Multipliers with Two Constraints

Example 4 (cont'd): We now have the system

$$\begin{aligned} x - y + z &= 1 \\ x^2 + y^2 &= 1 \\ -12x &= 5y \end{aligned}$$

First solve the second two equations for x, y, then find z using the first equation. There are two solutions for (x, y, z):

$$\left(\frac{5}{13},\ -\frac{12}{13},\ -\frac{4}{13}\right), \qquad \left(-\frac{5}{13},\ \frac{12}{13},\ \frac{30}{13}\right).$$

$f\left(\frac{5}{13},\ \frac{-12}{13},\ \frac{-4}{13}\right) = -6$: **absolute minimum**

$f\left(-\frac{5}{13},\ \frac{12}{13},\ \frac{30}{13}\right) = 20$: **absolute maximum**